Chapter 2: Limits and Continuity

2.1: Rates of Change and Tangents to Curves

Calculus helps us understand how the change in one quantity is related to the change in another quantity. 17^{th} Century - interested in study of motion of objects, particularly planets and stars. Need to know speed and direction at any instant. Need to find tangent to the path of motion at any given point. In this chapter, we discuss limits which are necessary for finding these tangent lines.

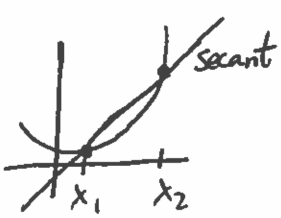
In 16^{th} Century, Galileo discovered that an object in free fall will fall a distance proportional to the square of the time it has been falling.

$$y = 16t^2 \qquad (y = \text{feet},\ t = \text{seconds})$$

The <u>average</u> <u>speed</u> over an interval is found by dividing the distance covered by the time elapsed. (feet per second).

<u>Average Rate of Change</u>: of $y = f(x)$ over $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \text{ where } h = x_2 - x_1.$$



Slope of Secant Line = Difference Quotient

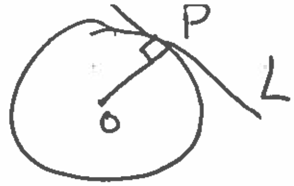
<u>Ex:</u> (1) A rock breaks free from a tall cliff. What is average speed

(a) in first 2 seconds (b) Second Second (32, 48 ft/sec)

(2) Approximate the speed of the rock at $t=1$ and $t=2$ seconds.

The tangent line to a curve is approximated by the secant lines. We call the slope of the tangent line at a point, the slope of the curve at that same point.

For circles:



The tangent line L, to the point P is perpendicular to the radius at P.

Ex: (3) Find the slope of the parabola $y=x^2$ at the point $P(2,4)$. Write an equation for the tangent line.

$P(2,4)$ $Q(2+h,(2+h)^2)$. Secant Slope $= h+4 \xrightarrow[h\to 0]{}$ tangent slope $=4$

Tangent line: $y=4x-4$

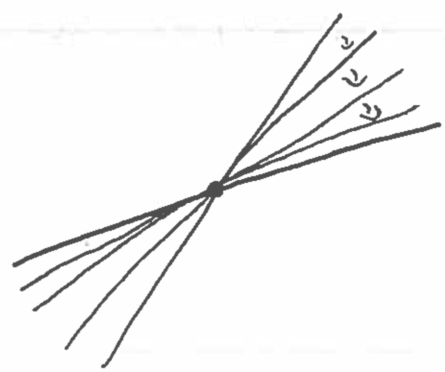
☆ Slope of Tangent Line = Instantaneous Rate of Change ☆

~~Limit of a Function and Limit Laws~~

Two ways to visualize:

(1)

(2)



Actually the same if the elements of the space in (1) are lines in $\mathbb{R}^2$.

2.2: Limit of a Function and Limit Laws

Here, we still treat limits intuitively. We give a precise definition in 2.3.

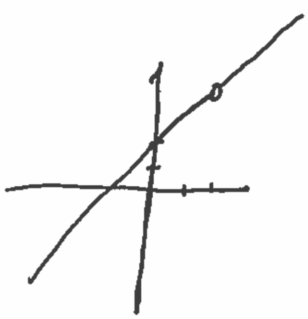
Frequently, when studying a fcn $y=f(x)$, we find ourselves interested in the behavior near a point c, instead of at that point. Sometimes this can give us more information. For instance, evaluating fcns at irrational values such as $\pi, \sqrt{2}$. (Ex. $2^{\sqrt{3}}$).

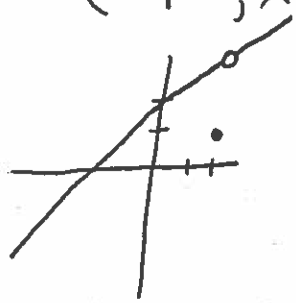
Another situation occurs when evaluating at c would lead to dividing by zero. (Ex. $\frac{x^2-4}{x-2}$ at $x=2$).

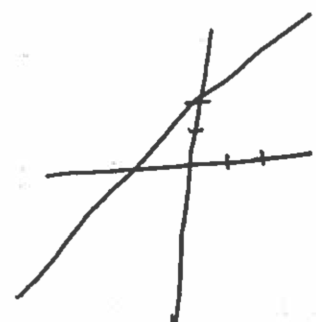
Or, more pertinent, in the difference quotient $\frac{f(x+h)-f(x)}{h} = \frac{\Delta Y}{K}$.

Our definition of close is still informal here and depends on context.

Ex: $f(x) = \frac{x^2-4}{x-2}$ $\qquad$ $g(x) = \begin{cases} \frac{x^2-4}{x-2}, & x \neq 2 \\ 1, & x=2 \end{cases}$ $\qquad$ $h(x) = x+2$





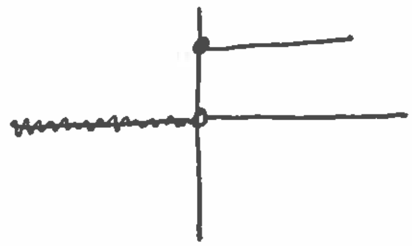


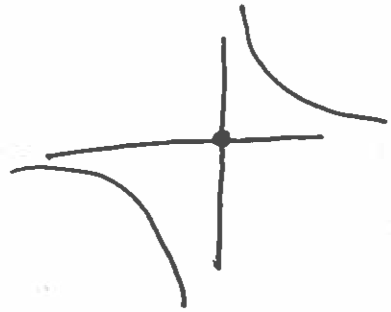
All three have different values at $x=2$, or undefined, but all have the same limit. So the three are essentially the same.

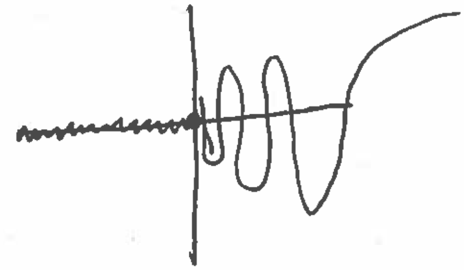
Ex: (a) $f(x) = x$; $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} x = c$

(b) $f(x) = K$; $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} K = K$.

Ex: (3) $y = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$ $\qquad y = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ $\qquad y = \begin{cases} 0, & x \leq 0 \\ \sin(\frac{1}{x}), & x > 0 \end{cases}$







All three have no limit at $x=0$.

However, we can discuss left and right hand limits.

Limit Laws: Let L, M, c and k be real numbers. (Limit Constant)

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M$$

(i) Sum/Difference: $\lim_{x \to c} f(x) \pm g(x) = L \pm M$

(ii) Constant Multiple: $\lim_{x \to c} k f(x) = k \cdot L$

(iii) Product: $\lim_{x \to c} f(x) \cdot g(x) = L \cdot M$

(iv) Quotient: $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M} \quad (M \neq 0)$

(v) Exponent: $\lim_{x \to c} [f(x)]^n = L^n \quad (n \in \mathbb{R})$

Ex: (4) Find $\lim_{x \to c} (x^3 + 4x^2 - 3)$, $\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5}$, $\lim_{x \to -2} \sqrt{4x^2 - 3}$

Theorem 1: Limits of Polynomials

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \Rightarrow \lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c + a$$

Theorem 2: Limits of Rationals

$P(x)$ and $Q(x)$ are polynomials such that $Q(c) \neq 0$. Then

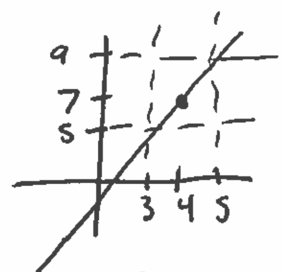
$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Theorem 4: If $f(x) \le g(x)$ for all x is some "small" interval around c, ~~then~~ except possibly at c, then

$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x).$$

2.3: Precise Definition of a Limit

We replace "close" and "small" with real definitions. ε and δ will always be greater than 0.



Ex: (1) $y = 2x-1$ near $x=4$

Intuitively, $\lim_{x\to 4} 2x-1 = 7$.

How close to $x=4$ must we be in order for $y=2x-1$ to be within 2 units of 7?

$$|y-7|<2 \Rightarrow |2x-1-7|<2 \Rightarrow |2x-8|<2$$

So $\quad -2 < 2x-8 < 2$

$\quad 6 < 2x < 10$

$\quad 3 < x < 5$. we must be within 1 of $x=4$.

Why 2 units above? No reason, we could have done any value.

Def^n: Let $f(x)$ be defined on some interval containing c, but not necessarily at c. Then the <u>limit of $f(x)$ at c is L</u>

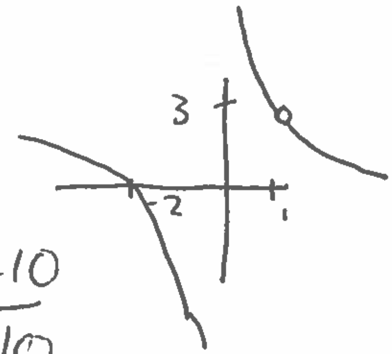
$$\lim_{x\to c} f(x) = L$$

if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{if} \quad 0 < |x-c| < \delta \quad \text{then} \quad |f(x)-L| < \varepsilon.$$

Ex: (5) $\lim\limits_{x\to -1} \dfrac{x^3+4x^2-3}{x^2+5} = \dfrac{(-1)^5+4(-1)^2-3}{(-1)^2+5} = \dfrac{0}{6} = 0.$

Ex: (6) $\lim\limits_{x\to 1} \dfrac{x^2+x-2}{x^2-x} = \lim\limits_{x\to 1} \dfrac{(x+2)(x-1)}{x(x-1)} = \lim\limits_{x\to 1} \dfrac{x+2}{x} = 3.$



Ex (7): $\lim\limits_{x\to 0} \dfrac{\sqrt{x^2+100}-10}{x^2} = \lim\limits_{x\to 0} \dfrac{\sqrt{x^2+100}-10}{x^2} \cdot \dfrac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}$

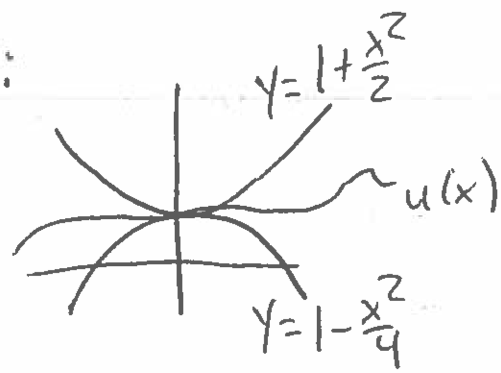
$= \lim\limits_{x\to 0} \dfrac{x^2+100-100}{x^2(\sqrt{x^2+100}+10)} = \lim\limits_{x\to 0} \dfrac{1}{\sqrt{x^2+100}+10} = \dfrac{1}{20}.$

The Sandwich Theorem: Suppose $g(x) \le f(x) \le h(x)$ for all x in a "small" interval around c. If

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$$

then $\lim\limits_{x\to c} f(x) = L.$

Ex (8):



If $1-\frac{x^2}{4} \le u(x) \le 1+\frac{x^2}{2}$ then $\lim\limits_{x\to 0} u(x) = 1.$

Ex (9): $-|x| \le \sin x \le |x|$ so $\lim\limits_{x\to 0} \sin x = 0$

$0 \le 1-\cos x \le |x|$ so $\lim\limits_{x\to 0} 1-\cos x = 0 \Rightarrow \lim\limits_{x\to 0} \cos x = 1$

If $\lim\limits_{x\to c} |f(x)| = 0$ then $\lim\limits_{x\to c} f(x) = 0.$

Ex: (2) ~~[scribble]~~ Find limit of $f(x) = 5x-3$ at $x=2$.

Let $\varepsilon > 0$. Need to find $\delta > 0$.

$$|f(x) - 2| < \varepsilon. \Rightarrow |5x-5| < \varepsilon$$

$$-\varepsilon < 5x - 5 < \varepsilon$$

$$5-\varepsilon < 5x < 5+\varepsilon$$

$$1-\frac{\varepsilon}{5} < x < 1+\frac{\varepsilon}{5} \Rightarrow \delta = \frac{\varepsilon}{5} \text{ works. (Can use smaller)}$$

Ex (3) (a) $\lim_{x\to c} x = c$. Let $\varepsilon > 0$. $|x-c| < \varepsilon$

$$-\varepsilon < x - c < \varepsilon$$

$$c-\varepsilon < x < c+\varepsilon \Rightarrow \delta = c.$$

(b) $\lim_{x\to c} k = k$. Let $\varepsilon > 0$. $|k-k| < \varepsilon$.

$$-\varepsilon < 0 < \varepsilon \Rightarrow \delta \text{ can be anything.}$$

If you want to show $\lim_{x\to c} f(x) = L$, you do two things.

(1) Solve $|f(x) - L| < \varepsilon$

(2) Find $\delta > 0$, such that if $x \in (c-\delta, c+\delta)$ then $|f(x) - L| < \varepsilon$.

Ex: (4) Show $\lim_{x\to 5} \sqrt{x-1} = 2$. Let $\varepsilon > 0$.

(1) $|\sqrt{x-1} - 2| < \varepsilon \Rightarrow 2-\varepsilon < \sqrt{x-1} < 2+\varepsilon \Rightarrow (2-\varepsilon)^2 < x-1 < (2+\varepsilon)^2$

$$(2-\varepsilon)^2 + 1 < x < (2+\varepsilon)^2 + 1 \Rightarrow 5 - 4\varepsilon + \varepsilon^2 < x < 5 + 4\varepsilon + \varepsilon^2.$$

$$\Rightarrow 5 - (4\varepsilon - \varepsilon^2) < x < 5 + (4\varepsilon + \varepsilon^2)$$

So $\delta = \min(4\varepsilon - \varepsilon^2, 4\varepsilon + \varepsilon^2) = 4\varepsilon - \varepsilon^2$ as long as $\varepsilon < 4$.

Ex: (5) Show $\lim_{x\to 2} f(x)=4$ where $f(x)=\begin{cases} x^2, & x\neq 2 \\ 1, & x=2. \end{cases}$

Let $\varepsilon>0$. $|f(x)-4|<\varepsilon \Rightarrow \sqrt{4-\varepsilon}<x<\sqrt{4+\varepsilon}$

So $\delta=\min(2-\sqrt{4-\varepsilon}, 2-\sqrt{4+\varepsilon})$.

From the definition, it is almost immediate that the theorems from the last section are true.

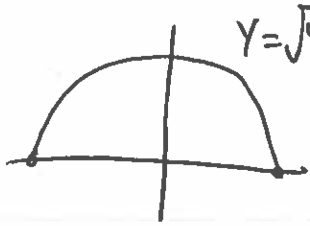
2.4: One-Sided Limits

So far we have discussed two-sided limits. One-sided limits are defined the same way except we ignore one side.

Right: $\lim_{x\to c^+} f(x)=L$ Left: $\lim_{x\to c^-} f(x)=L$.

These are most useful for endpoints of the domain.

Ex(1):

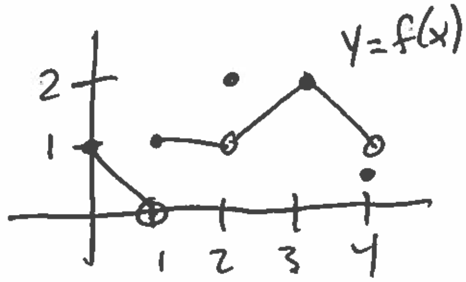


$\lim_{x\to -2^+} \sqrt{4-x^2}=0$ $\lim_{x\to 2^-} \sqrt{4-x^2}=0$

Theorem: A fcn has a two-sided limit if and only if it has a left and right limit and they agree.

$$\lim_{x\to c} f(x)=L \iff \lim_{x\to c^-} f(x)=L \text{ and } \lim_{x\to c^+} f(x)=L.$$

Ex: (2)



x	Left	Right
0	DNE	1
1	0	1
2	1	1
3	2	2
4	1	DNE

Ex: (3) Show $f(x)=\sin(1/x)$ has no limit at $x=0$.

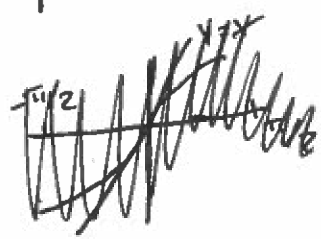
$\sin\left(\frac{1}{x}\right)=1$ whenever $\frac{1}{x}=\frac{\pi}{2}+2\pi n$ for some $n\in\mathbb{Z}$

So when $x=\dfrac{1}{\pi/2+2\pi n}$. But as $n\to\infty$, $x\to 0$. So when $\varepsilon=1$, no $\delta>0$ will work.

Theorem (7): $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$ Sandwich Theorem between

~~Sandwich theorem,~~

~~beta~~



$\cos\theta<\dfrac{\sin\theta}{\theta}<1$

when $\theta\in(0,\frac{\pi}{2})$.

Ex: (4) (a) $\displaystyle\lim_{h\to 0}\frac{\cos h-1}{h}=0$; $\dfrac{\cos h-1}{h}=\dfrac{1-2\sin^2(h/2)-1}{h}=-\dfrac{2\sin(h/2)}{h/2}\cdot\sin(h/2)$

So $\displaystyle\lim_{h\to 0}\frac{\cosh-1}{h}=(-1)(0)=0$. (half-angle formula)

(b) $\displaystyle\lim_{x\to 0}\frac{\sin(2x)}{5x}=\lim_{x\to 0}\frac{(2/5)\sin(2x)}{(2/5)5x}=\frac{2}{5}\lim_{x\to 0}\frac{\sin(2x)}{2x}=\frac{2}{5}(1)=\frac{2}{5}$

(c) $\displaystyle\lim_{t\to 0}\frac{\tan t\,\sec(2t)}{3t}=\lim_{t\to 0}\frac{1}{3}\,\frac{\sin t}{t}\cdot\frac{1}{\cos t}\cdot\frac{1}{\cos 2t}=\frac{1}{3}(1)(1)(1)=\frac{1}{3}$.

2.5: Continuity

A fcn is <u>continuous</u> at c if $\lim_{x \to c} f(x) = f(c)$

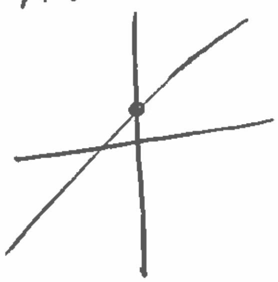
<u>right-cont</u> $\lim_{x \to c^+} f(x) = f(c)$

<u>left-cont</u> $\lim_{x \to c^-} f(x) = f(c)$.

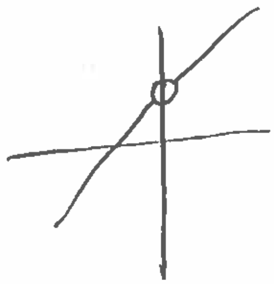
Otherwise we say f is <u>discontinuous</u> at c.

<u>Ex</u>: (1) $f(x) = \lfloor x \rfloor$ is not continuous at $n \in \mathbb{Z}$.

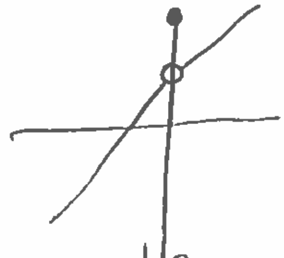
At $x=0$



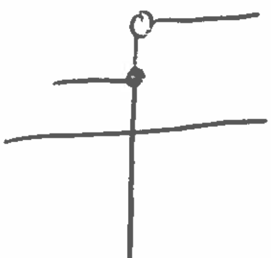
continuous



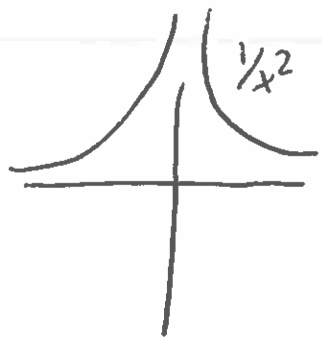
Dis continuous



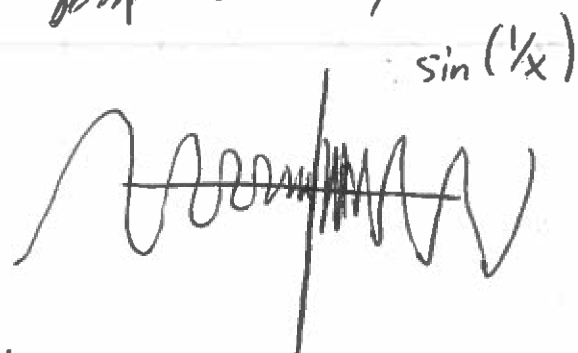
Removable ~~jump~~ discontinuity



jump discontinuity



infinite discontinuity



Oscillating discont.

<u>Def</u>: A fcn which is continuous at every point in its ~~on its~~ domain is called a <u>Continuous</u> <u>fcn</u>.